MATH4800 - FALL '23 - SUGGESTED EXERCISES 1

Problem 1. Show that Ω is a compact set with the metric $d(x,y) = 2^{-N(x,y)}$, where

$$N(x,y) = \max \{n \in \mathbb{Z} : x_k = y_k \text{ for all } k \leq n\}.$$

Problem 2. Let $f : X \to X$ be a dynamical system. Show that an point is periodic if and only if it is finite. If x is a periodic point, show that the set $S(x) = \{n \in \mathbb{Z} : f^n(x) = x\}$ is equal to $p\mathbb{Z}$ for a unique $p > 0$. Describe the periodic orbits of σ on Ω.

Problem 3. Let A be a 0-1 square $n \times n$ matrix. Show that Ω_A is closed by showing that $\Omega \setminus \Omega_A$ is open.

Problem 4. Find matrices A such that:

(1) Ω_A is empty
(2) Ω_A is finite
(3) Ω_A is countable but not finite
(4) Ω_A is uncountable

and justify your answer, or prove that no such matrix exists.

Problem 5. Fix the alphabet $\mathcal{A} = \{1,2,3\}$. Find the transition matrix A whose corresponding subshift Ω_A is described by the following:

(1) the sum of adjacent letters in the word is always odd
(2) the product of adjancent letters in the word is always even
(3) the letters 1 and 2 are always followed by larger letters

Problem 6 (Hard)**.** If $\mathcal{A} = \{1,2,\dots,n\}$ is any alphabet and ℓ is any positive integer, define a new alphabet $\mathcal{A}_2$ to be all words of length 2 in $\mathcal{A}$. Embed the full shift on $\mathcal{A}$ into the shift on $\mathcal{A}_2$ as a shift-invariant subset, and show that it is Ω_A for some $n^2 \times n^2$ matrix A. Show that the subshift which has rules involving the admissibility or banning of words of length at most 3 can always be realized as a subshift which allows or bans words of length 2 by increasing the alphabet size.

A subset Ω which bans certain words of a given, bounded length is called a *subshift of finite type*.

Problem 7 (Hard)**.** Find an example of a closed, σ-invariant subset of Ω which is not a subshift of finite type, and prove it is not a subshift of finite type. [*Hint*: Try to find a condition on words in Ω which cannot be detected by looking at finite strings at a time. Remember, a key difficult will be showing the closed property.]